به نام خدا

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۱ - الف)

گسترش زوج

$f(x) = x \quad [0,1] \quad L=1$

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\pi x$$

$$a_0 = \frac{1}{1}\int_0^1 f(x)\,dx = \frac{1}{2}$$

$$a_k = \frac{2}{1}\int_0^1 f(x)\cos k\pi x\,dx = 2\int_0^1 x\cos k\pi x\,dx$$

$$= 2\left[\frac{x\sin k\pi x}{k\pi}\Big)_0^1 - \int_0^1 \frac{\sin k\pi x}{k\pi}\,dx\right]$$

$$= 2\,\frac{\cos k\pi x}{k^2\pi^2}\Big)_0^1 = 2\,\frac{(-1)^k - 1}{k^2\pi^2} = \begin{cases} 0 & k \text{ زوج} \\ \dfrac{-4}{k^2\pi^2} & k \text{ فرد} \end{cases}$$

$$x = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi^2}\cos(2k-1)\pi x$$

ب)

@ $x=0$ تابع پیوسته $\longrightarrow$ $0 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi^2} \times 1$

$$\longrightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} = S_{odd}$$

هر دو درست

است.

@ $x=1$ $\longrightarrow$ $1 = \frac{1}{2} + \sum \frac{-4}{(2k-1)^2\pi^2}\overbrace{\cos(2k-1)\pi}^{=-1}$

$$\longrightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} = S_{odd}$$

$$S_{even} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}S$$

$$S = S_{odd} + \cancel{S_{even}}$$

$$\frac{1}{4}S$$

$$\frac{3}{4}S = S_{odd} = \frac{\pi^2}{8} \longrightarrow$$

$$S = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

$$S_{even} = \frac{1}{4}S = \frac{\pi^2}{24}$$

$u_t = u_{xx} + 4x$

$u(0,t) = 0$

$u(1,t) = t$

$u(x,0) = 1$

$v(x,t) = u(x,t) + a(t)x + b(t)$

$v(\cancel{0},t) = u(\cancel{0},t) + b(t) \longrightarrow b(t) = 0$

$v(\cancel{1},t) = u(\cancel{1},t) + a(t) \longrightarrow a(t) = -t$

$v(x,t) = u(x,t) - xt$

$v_t = u_t - x$

$v_{xx} = u_{xx}$

$v_t + x = v_{xx} + 4x \longrightarrow v_t = v_{xx} + 3x$

$v(0,t) = v(1,t) = 0$

$v(x,0) = u(x,0) = 1$

گام ۱

$v_t = v_{xx}$

$v(0,t) = v(1,t) = 0$ $\quad v = XT \quad XT' = X''T$

$$\frac{T'}{T} = \frac{X''}{X} = \alpha$$

I) $\alpha > 0 \quad \alpha = \lambda^2 \quad \frac{X''}{X} = \lambda^2 \longrightarrow X(x) = ae^{\lambda x} + be^{-\lambda x}$

$X(0) = X(1) = 0$

$\downarrow$

$a + b = 0 \longrightarrow b = -a$

$X(1) = ae^{\lambda} - ae^{-\lambda} = 0 \longrightarrow a = 0 \longrightarrow u = 0$ ✗

$\searrow \lambda = 0$ ✗

II) $\alpha = 0$

$X'' = 0 \longrightarrow X(x) = ax + b \quad X(0) = X(1) = 0 \longrightarrow a = b = 0$ ✗

III) $\alpha < 0 \qquad \alpha = -\lambda^2$

$$\frac{X''}{X} = -\lambda^2 \qquad X(x) = a\cos\lambda x + b\sin\lambda x$$

$$X(0) = a = 0$$

$$X(1) = b\sin\lambda = 0 \longrightarrow \lambda_n = n\pi$$

$$\boxed{X_n(x) = \sin n\pi x}$$

$$3x = \sum_{n=1}^{\infty} A_n \sin n\pi x$$ (گام ۲

$$A_n = \frac{2}{1}\int_0^1 3x\sin n\pi x\,dx = 6\int_0^1 x\sin n\pi x\,dx$$

$$= -\frac{6x\cos n\pi x}{n\pi} + \int_0^1 \frac{6\cos n\pi x}{n\pi}\,dx$$

$$= -\frac{6x\cos n\pi x}{n\pi}\Big]_0^1 + \frac{6\sin n\pi x}{n^2\pi^2}\Big]_0^1$$

$$= -\frac{6(-1)^n}{n\pi} = \frac{6(-1)^{n+1}}{n\pi}$$

$$v(x,t) = \sum X_n T_n = \sum \sin n\pi x\, T_n$$ (گام ۳

$$v_t = \sum \sin n\pi x\, T_n' \qquad v_{xx} = \sum -(n\pi)^2 \sin n\pi x\, T_n$$

$$\rightarrow \sum \sin n\pi x\, T_n' = \sum -(n\pi)^2 \sin n\pi x\, T_n + \sum \frac{6(-1)^{n+1}}{n\pi}\sin n\pi x$$

$$\rightarrow T_n' + (n\pi)^2 T_n = \frac{6(-1)^{n+1}}{n\pi}$$

عامل * $\quad \times e^{\int (n\pi)^2 dt} = e^{n^2\pi^2 t}$

$$\cancel{\left(T_n e^{(n\pi)^2 t}\right)' = \frac{6(-1)^{n+1}}{n\pi}} \longrightarrow T_n e^{n^2\pi^2 t} = \frac{6(-1)^{n+1}}{n\pi}t + C_n$$

$$T_n = k_n e^{-n^2\pi^2 t} + \frac{6(-1)^{n+1}}{n\pi} t e^{-n^2\pi^2 t}$$

$$\left(T_n e^{n^2\pi^2 t}\right)' = \frac{6(-1)^{n+1}}{n\pi} e^{n^2\pi^2 t}$$

$$T_n e^{n^2\pi^2 t} = \frac{6(-1)^{n+1}}{n^3\pi^3} e^{n^2\pi^2 t} + k_n$$

$$T_n = \frac{6(-1)^{n+1}}{n^3\pi^3} + k_n e^{-n^2\pi^2 t}$$

۳) ادامه * $T_n = k_n \times$ جواب عمومی همگن + جواب خصوصی

جواب همگن $T_n' + n^2\pi^2 T_n = 0 \longrightarrow \frac{T_n'}{T_n} = -n^2\pi^2$

$$\ln T_n = -n^2\pi^2 t + C$$

$$T_n = k_n e^{-n^2\pi^2 t}$$

جواب خصوصی $\tilde{T}_n = \alpha \longrightarrow 0 + n^2\pi^2\alpha = \frac{6(-1)^{n+1}}{n\pi} \longrightarrow \alpha = \frac{6(-1)^{n+1}}{n^3\pi^3}$

$$\Rightarrow T_n = k_n e^{-n^2\pi^2 t} + \frac{6(-1)^{n+1}}{n^3\pi^3}$$

گام ۴)

$$v(x,t) = \sum X_n T_n$$

$$= \sum_{n=1}^{\infty} \left(k_n e^{-n^2\pi^2 t} + \frac{6(-1)^{n+1}}{n^3\pi^3}\right) \sin n\pi x$$

گام ۵)

$$v(x,0) = 1 = \sum_{n=1}^{\infty} \left(k_n + \frac{6(-1)^{n+1}}{n^3\pi^3}\right) \sin n\pi x$$

$$k_n + \frac{6(-1)^{n+1}}{n^3\pi^3} = 2\int_0^1 \sin n\pi x \, dx = -\frac{2\cos n\pi x}{n\pi} = \frac{1-(-1)^n}{n\pi} \times 2$$

$$\rightarrow k_n = \frac{2-2(-1)^n}{n\pi} - \frac{6(-1)^{n+1}}{n^3\pi^3}$$

* توجه - به بعضی از دانشجویان که به مشکل ناپیوستگی در $x=0$ توجه کردند، توصیه شده که با $V(x,0)=0$ حل کنند بنابراین حل زیر برای گام ۵ هم نمره کامل می‌گیرد:

$V(x,0)=0 \longrightarrow$

$$k_n + \frac{6(-1)^{n+1}}{n^3\pi^3} = 0 \longrightarrow k_n = \frac{6(-1)^n}{n^3\pi^3}$$

$\longleftarrow$ با جایگذاری k_n ، V ثابت می‌شود $\longleftarrow$ $u = V + xt$

-۳

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

-گام ۱

$$u=R\Theta$$

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0$$

$$r^2R''\Theta+rR'\Theta+R\Theta''=0 \qquad /R\Theta$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\alpha \longrightarrow \begin{cases}\dfrac{\Theta''}{\Theta}=-\alpha\\ r^2\dfrac{R''}{R}+r\dfrac{R'}{R}=\alpha\end{cases}$$

معادلی که شرایط مرزی همگن دارد

-گام ۲

$$\left.\begin{array}{l}\dfrac{\Theta''}{\Theta}=-\alpha\\ \Theta(0)=\Theta(\pi)=0\end{array}\right\}$$

شبیه مسائل ۱ $\longrightarrow$ $-\alpha<0$ $\quad \alpha>0$ $\quad \alpha=\lambda^2$

$$\Theta_n(\theta)=?$$

$$\Theta(\theta)=a\cos\lambda\theta+b\sin\lambda\theta$$

$$\Theta(0)=a=0 \qquad \Theta(\pi)=b\sin\lambda\pi=0 \longrightarrow \lambda\pi=n\pi$$

$$\boxed{\lambda_n=n} \qquad \alpha_n=n^2$$

-گام ۳

$$r^2R''+rR'-\underbrace{\alpha}_{n^2}R=0 \xrightarrow{\text{روش اویلر}} s^2+(1-1)s-n^2=0$$

$$s=\pm n \qquad r^n,\ r^{-n}$$

$$R_n=A_nr^n+B_nr^{-n}$$

$$\left.\begin{array}{l}u<\infty\\ r\to 0\end{array}\right\} \quad B_n=0 \qquad R_n(r)=A_nr^n$$

-گام ۴

$$u(r,\theta)=\sum_{n=1}^{\infty}A_nr^n\sin n\theta$$

$$u(2,\theta) = \theta^2 - \pi\theta = \sum_{n=1}^{\infty} A_n 2^n \sin n\theta$$ - ص ۳۶

$$A_n 2^n = \frac{2}{\pi}\int_0^{\pi} (\theta^2 - \pi\theta)\sin n\theta \, d\theta$$

$$I_1 = \int_0^{\pi} \theta^2 \sin n\theta \, d\theta = -\frac{\theta^2 \cos n\theta}{n} + \int_0^{\pi} \frac{2\theta \cos n\theta}{n} d\theta$$

$$= -\frac{\pi^2(-1)^n}{n} + \frac{2}{n}\left[\cancel{\frac{\theta \sin n\theta}{n}}\Big|_0 - \int_0^{\pi} \frac{\sin n\theta}{n}\right]$$

$$= \frac{\pi^2(-1)^{n+1}}{n} + \frac{2}{n}\frac{\cos n\theta}{n^2}\Big]_0^{\pi} = \frac{n^2\pi^2(-1)^{n+1} + 2(-1)^n - 2}{n^3}$$

$$I_2 = \int_0^{\pi} \theta \sin n\theta \, d\theta = -\frac{\theta \cos n\theta}{n}\Big]_0^{\pi} + \int_0^{\pi} \frac{\cos n\theta}{n} d\theta$$

$$= -\frac{\pi(-1)^n}{n} = \frac{\pi(-1)^{n+1}}{n} + 0$$

$$\Rightarrow 2^n A_n = \frac{2}{\pi}\left[I_1 - \pi I_2\right] = \frac{2}{\pi}\left[\frac{\cancel{n^2\pi^2(-1)^{n+1}} + 2(-1)^n - 2 - \cancel{n^2\pi^2(-1)^{n+1}}}{n^3}\right]$$

$$= \frac{4(-1)^n - 4}{n^3\pi} \longrightarrow A_n = \frac{4(-1)^n - 4}{n^3 \pi 2^n}$$

ع -

$u_{tt} = u_{xx} + u_{yy}$

$u = XYT$ گام ۱

$$XYT'' = X'YT + XY''T \qquad /XYT$$

$$\frac{T''}{T} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{} = \alpha \qquad \frac{X''}{X} = \beta$$

$$\frac{X''}{X} = \alpha - \frac{Y''}{Y} = \beta \qquad \frac{Y''}{Y} = \alpha - \beta = \gamma$$

$$\frac{T''}{T} = \alpha$$

گام ۲

$$\left.\begin{array}{l} \frac{X''}{X} = \beta \\ X(0) = X(1) = 0 \end{array}\right\} \quad \text{مسئله ۱} \qquad \beta < 0 \qquad \beta = -\lambda^2$$

$$X = a\cos\lambda x + b\sin\lambda x$$

$$X(0) = a = 0 \qquad X(1) = \sin\lambda = 0 \qquad \lambda = n\pi$$

$$\beta = -(n\pi)^2$$

$$X_n = \sin n\pi x$$

گام ۳

$$\left.\begin{array}{l} \frac{Y''}{Y} = \gamma \\ Y(0) = Y(2) = 0 \end{array}\right\} \quad \gamma < 0 \qquad Y_m(y) = \sin\frac{m\pi}{2}y \qquad \gamma = -\left(\frac{m\pi}{2}\right)^2$$

گام ۴

$$\frac{T''}{T} = \alpha = \beta + \alpha - \beta = \beta + \gamma = -\left((n\pi)^2 + \left(\frac{m\pi}{2}\right)^2\right) = -\mu_{mn}^2$$

$$T_{mn} = A_{mn}\cos\mu_{mn}t + B_{mn}\sin\mu_{mn}t$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\cos \mu_{mn} t + B_{mn}\sin \mu_{mn} t\right) \sin n\pi x \sin \frac{m\pi}{2} y$$ ω ρb

$$u(x,y,0) = \sum_{m=1}^{\infty}\underbrace{\sum_{n=1}^{\infty} A_{mn}\sin n\pi x}_{A_m(x)} \sin \frac{m\pi}{2} y = \sin \pi x \sin \pi y$$ 4 ρb

$$A_m(x) = \frac{2}{2}\int_0^2 \sin\pi x \sin \pi y \sin \frac{m\pi}{2} y \, dy$$

$$= \sin \pi x \int_0^2 \sin \pi y \sin \frac{m\pi}{2} y \, dy$$

$$\overset{(m\neq 2)}{=} \sin \pi x \int_0^2 \frac{1}{2}\left(\cos\left(\pi - \frac{m\pi}{2}\right)y - \cos\left(\pi + \frac{m\pi}{2}\right)y\right) dy$$

$$= 0$$

@ $m=2$ $$A_m(x) = \sin \pi x \int_0^2 \sin^2 \pi y \, dy$$

$$= \sin\pi x \int_0^2 \frac{1 - \cos 2\pi y}{2} dy = \sin \pi x$$

$$\sum A_{mn}\sin n\pi x = A_m(x) = \begin{cases} 0 & m\neq 2 \\ \sin \pi x & m=2 \end{cases}$$

$$A_{mn} = 2\int_0^1 A_m(x)\sin n\pi x \, dx = \begin{cases} 0 & m\neq 2 \\ 2\int_0^1 \sin\pi x \sin n\pi x \, dx & m=2 \end{cases}$$

$$\int_0^1 \sin \pi x \sin n\pi x \, dx \overset{(n\neq 1)}{=} \int_0^1 \frac{1}{2}\left(\cos(\pi - n\pi)x - \cos(\pi+n\pi)x\right) dx = 0$$

@ $n=1$ $$\int_0^1 \sin^2 \pi x \, dx = \int_0^1 \frac{1-\cos 2\pi x}{2} dx = \frac{1}{2} \Rightarrow$$

$$A_{mn} = \begin{cases} 0 & m \neq 2 \\ 0 & m = 2 \quad n \neq 1 \\ 1 & m = 2 \quad n = 1 \end{cases}$$

$$u_t(x,y,0) = \sum \sum \mu_{mn} B_{mn} \sin n\pi x \sin \frac{m\pi y}{2} = 0 \longrightarrow B_{mn} = 0$$

$$u(x,y,t) = \cos \mu_{21} t \sin \pi x \sin \pi y = \cos \sqrt{2}\pi t \sin \pi x \sin \pi y$$

$$\mu_{21} = \sqrt{\pi^2 + \pi^2} = \sqrt{2}\,\pi$$

$-\infty < x < \infty \longrightarrow$ [illegible] $U(\omega,t) = \mathcal{F}\{u(x,t)\}$ ه-

$\mathcal{F}\{u_t\} = U_t$

$\mathcal{F}\{u_{xx}\} = (i\omega)^2 U = -\omega^2 U$

$U_t + \omega^2 U = 0 \longrightarrow \dfrac{U_t}{U} = -\omega^2 \qquad \ln U = -\omega^2 t + c$

$$U(\omega,t) = C(\omega)\, e^{-\omega^2 t}$$

$$U(\omega,0) = \mathcal{F}\{u(x,0)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega x}}{-i\omega} \right)_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right) = \frac{1}{\sqrt{2\pi}} \frac{\cancel{i}2\sin\omega}{\cancel{i}\omega}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}$$

$$\longrightarrow U(\omega,t) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}\, e^{-\omega^2 t}$$

$$u(x,t) = \mathcal{F}^{-1}\{U\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, U\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, e^{-\omega^2 t}\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega\; e^{i\omega x - \omega^2 t}}{\omega}\, d\omega$$